

## Natural Units in Particle Physics

- kg m s or g cm s etc. are convenient for "everyday" objects.
- We are talking about sub-atomic particles, after all. So the usual units are not convenient.  
E.g.,  $m_{\text{proton}} = 1.67 \times 10^{-24} \text{ g}$
- So, we use units that are natural and convenient for the scale involved
- Size : 1 fermi  $\equiv$  fm  $= 10^{-15} \text{ m}$  or  $10^{-13} \text{ cm}$
- Energy: 1 eV  $= 1.6 \times 10^{-19} \text{ Joules}$   
(An electron volt is the energy gained by an electron or proton across 1 volt potential difference.)

Normally, higher units are used

$$\text{keV} = 10^3 \text{ eV}$$

$$\text{MeV} = 10^6 \text{ eV}$$

$$\text{GeV} = 10^9 \text{ eV}$$

$$\text{TeV} = 10^{12} \text{ eV}$$

Mass of the electron  $= .511 \text{ MeV}/c^2$  (or simply  
 $.511 \text{ MeV})$

Mass of the proton  $\approx 1 \text{ GeV}/c^2$

Tevatron beam energy  $\sim 1 \text{ TeV}$

## Natural Units (contd.)

- Also, we adopt the fundamental constants  $\hbar$  (the unit of action from QM) and  $c$  (the speed of light, Relativity) as fundamental units
- Some common quantities and dimensions

Energy	$\text{GeV}$	Time	$(\text{GeV}/\hbar)^{-1}$
Momentum	$\text{GeV}/c$	Length	$(\text{GeV}/\hbar c)^{-1}$
Mass	$\text{GeV}/c^2$	Area	$(\text{GeV}/\hbar c)^{-2}$
$[ M = E/c^2, L = \hbar c/E, T = \hbar/E ]$			

- We set  $\hbar = 1$ ,  $c = 1$  to simplify Algebra (and units)

Energy	$\text{GeV}$	Time	$\text{GeV}^{-1}$
Momentum	$\text{GeV}$	Length	$\text{GeV}^{-1}$
Mass	$\text{GeV}$	Area	$\text{GeV}^{-2}$

To convert back to normal units we need to restore the missing factors of  $\hbar$  and  $c$ .

## Natural Units (contd.) and useful conversions

- $\hbar c = 197 \text{ MeV-fm}$  ;  $\hbar = 6.582 \times 10^{-27} \text{ MeV-s}$
- Fine Structure Constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{e^2}{4\pi} \quad \left( \begin{array}{l} \text{Here } \epsilon_0 = 1, \text{ also} \\ \Rightarrow \mu_0 = 1 \text{ since} \\ c = 1/\epsilon_0 \mu_0 \end{array} \right)$$

$$= \frac{1}{137} \quad \left( \begin{array}{l} \text{so } e = 0.303 \\ = 1.6 \times 10^{-19} \text{ C} \end{array} \right)$$

- Cross Section (A measure of interaction or production rate)

$$1 \text{ barn} = 10^{-24} \text{ cm}^2$$

$$1 \text{ mb} = 1 \text{ millibarn} = 10^{-3} \text{ barn}$$

$$1 \mu\text{b} = 1 \text{ microbarn} = 10^{-6} \text{ barn}$$

$$1 \text{ nb} = 1 \text{ nanobarn} = 10^{-9} \text{ barn}$$

$$1 \text{ pb} = 1 \text{ picobarn} = 10^{-12} \text{ barn}$$

$$1 \text{ fb} = 1 \text{ femtobarn} = 10^{-15} \text{ barn}$$

E.g.,  $\sigma(t\bar{t})$  at  $\sqrt{s} = 2 \text{ TeV} \sim 7 \text{ pb}$

Total inelastic cross section  $\sim 60 \text{ mb}$

- As an aside collider luminosity  
 $\rightarrow \text{cm}^{-2} \text{ or } \text{pb}^{-1} \text{ or } \text{fb}^{-1}$

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## Non-relativistic QM

We need relativistic QM for particle physics, but let us start with the non-relativistic formulation.

Non-relativistic energy,

$$E = T + V = \frac{p^2}{2m} + V$$

Replace by operators,

$$\vec{p} \rightarrow -i\hbar \nabla ; \quad E \rightarrow i\hbar \frac{\partial}{\partial t}$$

Set  $V=0$  (free particle)

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial \psi}{\partial t}$$

Non-relativistic  
time-dependent  
Schrodinger Eqn.

Set  $\hbar=1, c=1$

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi$$

Plane wave solution:  $\psi(\vec{r}, t) = N e^{i(\vec{p} \cdot \vec{r} - Et)}$

Note: Schrodinger Eqn. is first order in time derivative and second order in space derivative

# Relativistic Schrödinger Equation or the Klein-Gordon Equation

Consider the relativistic case,

$$E^2 = p^2 c^2 + m^2 c^4$$

Again, set  $\vec{p} \rightarrow i\hbar \vec{\nabla}$ ;  $E \rightarrow i\hbar \frac{\partial}{\partial t}$

$$-\hbar^2 \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi(\vec{r}, t) + m^2 c^4 \cdot \psi(\vec{r}, t)$$

Set  $\hbar = 1$ ,  $c = 1$ , drop the coordinate variables, for simplicity,

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = (\nabla^2 - m^2) \psi}$$

K-G Equation

Using  $\partial_\mu = \left( \frac{\partial}{\partial t}, \nabla \right)$  and  $\partial^\mu = \left( \frac{\partial}{\partial t}, -\nabla \right)$ , and  $\square^2 = \partial_\mu \partial^\mu$   
 $(\square^2 + m^2) \psi = 0$   $\square = \text{D'Alembertian operator}$

For plane wave solutions,  $\psi = N e^{i(\vec{p} \cdot \vec{r} - Et)}$ ,

$$-E^2 \psi = -|\vec{p}|^2 \psi + m^2 \psi$$

$$\Rightarrow E = \pm \sqrt{p^2 + m^2}$$

$\therefore$  the K-G equation has negative energy solutions  
 More problematic  $\rightarrow$  it gives -ve particle densities  
 [These problems were later overcome in QFT] (5)

## The Dirac Equation

(Recap from earlier)

Schrödinger eqn.  $-\frac{1}{2m} \nabla^2 \psi = i \frac{\partial \psi}{\partial t}$  1<sup>st</sup> order in  $\frac{\partial}{\partial t}$   
 2<sup>nd</sup> order in  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

Klein-Gordon eqn.

$$(\partial^2 \psi + m^2) \psi = 0$$

or  $(\nabla^2 - m^2) \psi = \frac{\partial^2 \psi}{\partial t^2}$  2<sup>nd</sup> order in  $\frac{\partial}{\partial t}$   
 2<sup>nd</sup> order in  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

The problems with KG equation motivated Dirac to look for an alternate formulation of relativistic QM in which all particle densities are positive.

He wrote an equation that was first order in both space and time coordinates

$$H\psi = (\vec{\alpha} \cdot \vec{p} + \beta m) \psi = E\psi$$

$$\text{with } \vec{p} = -i \vec{\nabla}$$

$$(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m) \psi = i \frac{\partial \psi}{\partial t}$$

But, the solutions of this equation should also satisfy Klein-Gordon equation, in order to be a valid relativistic formulation of QM of a free particle.

Squaring both sides of the Dirac equation, one gets,

$$\begin{aligned}
 -\frac{\partial^2 \Psi}{\partial t^2} &= -\alpha_x^2 \frac{\partial^2 \Psi}{\partial x^2} - \alpha_y^2 \frac{\partial^2 \Psi}{\partial y^2} - \alpha_z^2 \frac{\partial^2 \Psi}{\partial z^2} + \beta^2 m^2 \Psi \\
 &- (\alpha_x \alpha_y + \alpha_y \alpha_x) \frac{\partial^2 \Psi}{\partial x \partial y} - (\alpha_y \alpha_z + \alpha_z \alpha_y) \frac{\partial^2 \Psi}{\partial y \partial z} - (\alpha_z \alpha_x + \alpha_x \alpha_z) \frac{\partial^2 \Psi}{\partial z \partial x} \\
 &- (\alpha_x \beta + \beta \alpha_x) m \frac{\partial \Psi}{\partial x} - (\alpha_y \beta + \beta \alpha_y) m \frac{\partial \Psi}{\partial y} - (\alpha_z \beta + \beta \alpha_z) m \frac{\partial \Psi}{\partial z}
 \end{aligned}$$

Comparing with K.G. equation

$$-\frac{\partial^2 \Psi}{\partial t^2} = -\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial z^2} + m^2 \Psi$$

$$\Rightarrow \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$

$$\left. \begin{aligned}
 \alpha_j \alpha_k + \alpha_k \alpha_j &= 0 \quad (j \neq k) \\
 \alpha_j \beta + \beta \alpha_j &= 0
 \end{aligned} \right\} \begin{matrix} \text{where indices} \\ x, y, z \text{ are replaced} \\ \text{by } j, k. \end{matrix}$$

If we replace the indices/subscripts  $\rightarrow 1, 2, 3$

and set  $\beta = \omega_e \cdot mc$

$$\alpha_a \alpha_b + \alpha_b \alpha_a = 2 \delta_{ab} \quad (a, b = 1, 2, 3 \text{ or } 4)$$

$$\delta_{ab} = 1 \text{ if } a = b$$

$$\delta_{ab} = 0 \text{ if } a \neq b$$

i.e., the four  $\alpha$ 's anticommute with one another and the square of each is unity.

If turns out that these  $\alpha$ 's (and  $\beta$ ) have to be  $4 \times 4$  matrices to satisfy the requirements.

So, the Dirac equation,

$$i \frac{\partial \psi}{\partial t} = -i \sum \alpha_i \frac{\partial \psi}{\partial x_i} + \beta \psi$$

can be written compactly as

$$(i \gamma^\mu \partial_\mu - m) \psi = 0$$

$\gamma^0 = \beta, \gamma^1 = \beta \alpha_1, \gamma^2 = \beta \alpha_2, \gamma^3 = \beta \alpha_3$ ; then multiply both sides by  $\beta$  and note  $\beta^2 = 1$

can be interpreted as a 4-dimensional matrix equation with four-component wave-function

$$\psi(\vec{r}, t) = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \\ \psi_3(\vec{r}, t) \\ \psi_4(\vec{r}, t) \end{pmatrix} \quad \text{Dirac Spinors}$$

Plane wave solutions take the form,

$$\psi(\vec{r}, t) = u(p) e^{i(\vec{p} \cdot \vec{r} - Et)}$$

where  $u(p)$  is also a 4-component spinor and satisfies the eigenvalue equation

$$H_p u(p) = (\vec{\alpha} \cdot \vec{p} + \beta m) u(p) = E u(p)$$

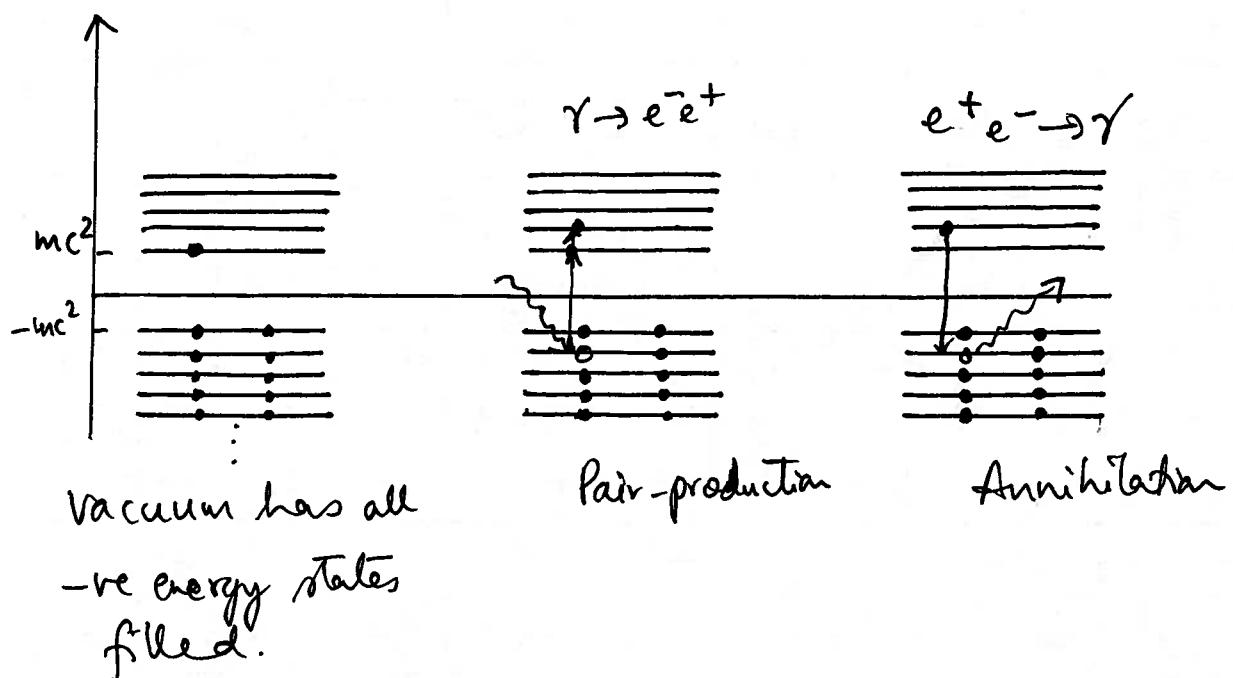
There are 4 solutions to this equation

- Two with +ve energy  $E = +E_p$  and two possible spin states for  $s = \frac{1}{2}$  particle
- Two with -ve energy  $E = -E_p$  and the two spin states

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## Dirac's interpretation of Negative energy states

In vacuum, all negative energy states are filled and so the Pauli exclusion principle prevents electrons falling into -ve energy states.



If an electron is added, it occupies one of the +ve energy states.

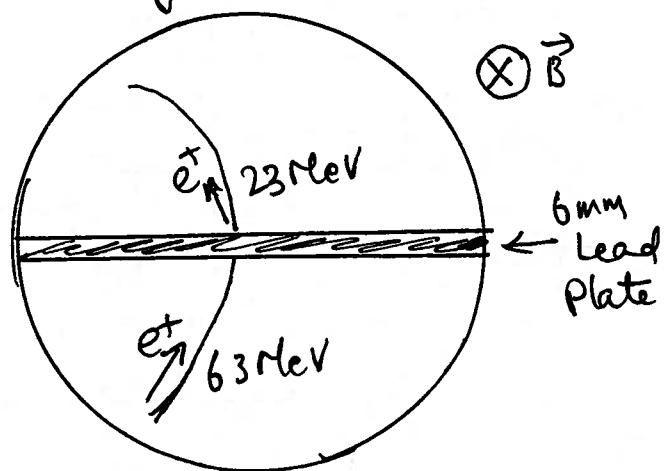
If an electron is removed from vacuum, this leaves a "hole"  $\leftarrow$  we are removing a negative energy electron,  $E = -E_p < 0$ , momentum  $-p$ , spin  $-S$  and charge  $-e$  from the vacuum. This means we have a hole with  $E = E_p > 0$ , positive  $p$ , spin  $S$  and charge  $+e$ .

Dirac boldly predicted the existence of a spin  $\frac{1}{2}$  particle with opposite charge ( $+e$ ) to that of electron with the same mass.

## Discovery of the Positron

C. D. Anderson, Phys. Rev. 43 (1933) 491

cosmic ray track in cloud chamber



- $e^+$  enters at bottom, showed down in the lead plate
- Curvature in the  $B$ -field shows that it is a positively charged particle.
- Can't be a proton  $\leftarrow$  would have stopped in lead.

So, the first anti-particle was discovered.  
And Dirac's theory was verified.

But, the interpretation of vacuum, etc.  
have been superseded by more sophisticated  
understanding and approach.

## Dirac's Equation (Contd.)

Dirac's equation and its solutions not only solved the problems with negative particle densities and predicted "antiparticles" based on the negative energy solutions, but lead to many other successful predictions.

One of the important predictions -  
the so-called "Dirac magnetic moment" for point-like spin- $\frac{1}{2}$  particles of mass  $m$  and charge  $q$

$$\vec{\mu}_D = q \vec{S} / m$$

(This intrinsic magnetic moment is twice that expected from a classical picture.  $\vec{\mu} = q \cdot \frac{g}{2m} \vec{S}$ )  
 $g = 2$ , gyromagnetic ratio

for proton and neutron, experimental measurements yield,

$$\vec{\mu}_p = 2.79 \frac{e \vec{S}}{m_p} \text{ and } \vec{\mu}_n = -1.91 \frac{e \vec{S}}{m_n}$$

→ indication that there are not point-like particles.